Lecture 3. Monoidal Transformation

Note Title 3/8/2020

X: smooth complex surface
$\overset{\cup}{p}$ $\rightsquigarrow$ $\pi : \tilde{X} \longrightarrow X$ blow-up at p
w/ exceptional curve E

Proposition 1: X projective/ Kähler
then $\tilde{X}$ smooth projective/ Kähler surface
w/ $E \cong \mathbb{P}^1$, $\underline{E^2} = -1$

$$\underline{E^2} \,\|\, \deg(\mathcal{N}_{E/\tilde{X}}) = \deg(\mathcal{O}(-1))$$

Proposition 2: (1) $\mathrm{Pic}(\tilde{X}) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}$

$$f^*D + nE \longleftarrow\!\shortmid (D, n)$$

(2) Intersection pairing on $\tilde{X}$

- $\pi^*C . \pi^*D = C.D$. $C, D \in \mathrm{Pic}(X)$
- $\pi^*C . E = 0$
- $E^2 = -1$
- $D \in \mathrm{Pic}(\tilde{X}) \xrightarrow{\pi_*} \mathrm{Pic}(X) \ni C$

  $\pi_* D . C = D . \pi^* C$

pf: (1) $\tilde{X} \setminus E \cong X \setminus \underline{p}$ $\Longrightarrow$ $0 \to \mathbb{Z}E \to \mathrm{Pic}(\tilde{X}) \underset{\curvearrowleft \pi^*}{\longrightarrow} \mathrm{Pic}(X) \to 0$

codim 2 — E irreducible of codim 1 — split

(2) C, D linear equivalent to differences of non-singular curves avoiding p

then (2) follows directly.

Proposition 3. $K_{\tilde{X}} = \pi^* K_X + E$. In particular, $K_{\tilde{X}}^2 = K_X^2 - 1$

pf: $\tilde{X} \setminus E \cong X \setminus P \Longrightarrow K_{\tilde{X}} = \pi^* K_X + rE$

adjunction formula $\Longrightarrow \cancel{2g}^{\,0}-2 = K_{\tilde{X}} + E\big|_E = (r+1)\underbrace{E^2}_{=-1}$

Proposition 4. $\pi_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_X$, $R^i \pi_* \mathcal{O}_{\tilde{X}} = 0$, $i > 0$

In particular, $H^i(X, \mathcal{O}_X) = H^i(\tilde{X}, \mathcal{O}_{\tilde{X}})$, $i \geq 0$

pf: $\tilde{X} \setminus E \cong X \setminus P \Longrightarrow \pi_* \mathcal{O}_X\big|_{X \setminus P} \cong \mathcal{O}_X\big|_{X \setminus P}$

$$\mathrm{Supp}\big(\underbrace{\boxed{R^i \pi_* \mathcal{O}_{\tilde{X}}}}_{= \mathcal{F}^i}\big) \subseteq P, \quad i > 0$$

$$\begin{tikzcd} E_n \arrow[r] \arrow[d, "\pi_n"'] \arrow[dr, phantom, "\square"] & \tilde{X} \arrow[d, "\pi"] \\ \mathrm{Spec}(\mathcal{O}_P / m_P^n) \arrow[r] & X \end{tikzcd}$$

$$\underbrace{(R^i \pi_* \mathcal{O}_{\tilde{X}})^{\wedge}_P}_{= \hat{\mathcal{F}}^i} \xrightarrow{\ \cong\ } \varprojlim H^i(E_n, \mathcal{O}_{E_n})$$

$\mathcal{J}$: ideal sheaf of E, then $\mathcal{J}^n$ ideal sheaf of E_n

$$0 \longrightarrow \mathcal{J}^n/\mathcal{J}^{n+1} \longrightarrow \underset{\substack{\cong \\ \mathcal{O}_{\tilde{X}}/\mathcal{J}^{n+1}}}{\mathcal{O}_{E_{n+1}}} \longrightarrow \underset{\substack{\cong \\ \mathcal{O}_{\tilde{X}}/\mathcal{J}^{n}}}{\mathcal{O}_{E_n}} \longrightarrow 0$$

since P is a smooth point $\mathcal{J}^n/\mathcal{J}^{n+1} \cong \mathrm{Sym}^n(\mathcal{J}/\mathcal{J}^2) \cong \mathcal{O}_E(n)$

$E_1 \cong \mathbb{P}^1$, $H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) = 0$, $\forall\, i > 0,\ n > 0$

By induction, $H^i(\mathcal{O}_{E_n}) = 0$, $\forall i > 0$

$\Longrightarrow \hat{\mathcal{F}}^i \cong \varprojlim H^i(E_n, \mathcal{O}_{E_n}) = 0$, $i > 0$.

$\Longrightarrow \mathcal{F}^i \cong \hat{\mathcal{F}}^i = 0$, since both only support on P.

Zariski's main theorem

- $f: X \longrightarrow Y$ birational projective between Noetherian integral schemes $\Longrightarrow \pi_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_X$
- Y normal

$\Longrightarrow f^{-1}(y)$ connected, $\forall y \in Y$

Leray spectral sequence

$$H^p(X, R^q\pi_*(\mathcal{O}_{\tilde{X}})) \Longrightarrow H^{p+q}(\tilde{X}, \mathcal{O}_{\tilde{X}})$$

implies that $H^i(X, \mathcal{O}_X) \cong H^i(\tilde{X}, \mathcal{O}_{\tilde{X}})$ //

Let $C \subseteq X$ effective divisor

$\tilde{C} \subseteq \tilde{X}$ strict transform of C

then $\pi^* C = \tilde{C} + r E$ and what is r?

Definition: $p \in X$. f local equation of C at p

$\mu_p(C) =$ largest $r \in \mathbb{N} \cup \{0\}$ s.t. $f^r \in m_p^r$

multiplicity of C at p

$\mu_P(C) \geq 1$ iff $C \ni P$

"$=$" $\quad P$ is a non-singular point.

Proposition 5. $\pi^* C = \tilde{C} + \mu_P(C) \cdot E$

pf: Write $r = \mu_P(C)$, then local equation f of C at P

$$f(x,y) = \underbrace{f_r(x,y)}_{\text{homog. degree } r \text{ polynomial in } x,y} + \text{higher order terms} \qquad \mathfrak{m}_P = (x,y)$$

$\tilde{X}$ locally given by equation

$$uy = vx \quad ; \quad (u:v) \in \mathbb{P}^1$$

In a neighborhood, $x=y=0$, $u \neq 0$

$\pi^* C$ has local equation

proper transform

$$f(x, \tfrac{v}{u}x) = x^r \big(\underline{f(1, \tfrac{v}{u}) + \text{higher order terms}} \big)$$

$x = 0 \underset{u \neq 0}{\Longrightarrow} y = 0$ thus $x=0$ is the local equation of E

Notice that
$$\begin{aligned} 2P_a(\tilde{C}) - 2 &= \tilde{C} . (\tilde{C} + K_{\tilde{X}}) \\ &= (\pi^* C - rE) . (\pi^* C - rE + \pi^* K_X + E) \\ &= C . (C+K) - r(r-1) \\ &= 2P_a(C) - 2 - r(r-1) \end{aligned}$$

or $\quad P_a(\tilde{C}) = P_a(C) - \frac{1}{2} r(r-1)$

proper transform/ blow up decrease the arithmetic genus

In particular $P_a(\tilde{C}) < P_a(C)$. if $r > 1$

i.e p singular point of C

Therefore, one can iteratively blow up a curve C in X and resolve its singularities.

The following theorem allows one to blow down (-1)-curves to obtain smooth surfaces.

Theorem (Castelnuovo) also true in the analytic category by Grauert.

X smooth projective complex surface

$\cup\!\!\!\shortmid$

$E \cong \mathbb{P}^1$, $E^2 = -1$

then $\exists$
$$\begin{array}{ccc} X & \longrightarrow & X_0 \\ \cup\!\!\!\shortmid & & \Psi \\ E & \longrightarrow & p \end{array}$$
monoidal transformation

X_0 smooth complex surface.

pf: (idea) construct a base point free linear system which contracts E & prove the image is smooth

hard part

Step 1: Choose H very ample divisor on X

s.t $H^1(X, \mathcal{O}_X(H)) = 0$ theorem of Serre

Let $k = H.E$, Claim: $H^1(X, \mathcal{O}_X(H+kE)) = 0$

we will actually prove $H^1(X, \mathcal{O}_X(H+iE)) = 0$, $i = 0, \cdots, k$ by induction

$$0 \longrightarrow \mathcal{O}_X(H+(i-1)E) \longrightarrow \mathcal{O}_X(H+iE) \longrightarrow \mathcal{O}_{\underset{\underset{\mathbb{P}^1}{\cong}}{E}}(k-i) \longrightarrow 0$$

$$\underbrace{H^1(X,\mathcal{O}_X(H+(i-1)E))}_{\substack{\parallel \\ 0}\ \text{induction hypothesis}} \longrightarrow H^1(X,\mathcal{O}_X(H+iE)) \longrightarrow \underbrace{H^1(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(k-i))}_{\substack{\parallel \\ 0}\ i\leq k}$$

Step 2. $|H+kE|$ is generated by global sections

- $|H+kE|$ is generated by global sections away from E.
- $$\begin{tikzcd} H^0(X,\mathcal{O}_X(H+kE)) \arrow[r] & H^0(E,\mathcal{O}_E) \arrow[r] & \underbrace{H^1(X,\mathcal{O}_X(H+(k-1)E))}_{\substack{\parallel \\ 0}\ \text{step 1}} \\ S \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & 1 \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

  generated global sections

S generates global sections near E

Thus,
$$\begin{tikzcd} X \arrow[r, "|H+kE|"] & \mathbb{P}^* \\ E \arrow[u, phantom, "\subseteq" rotate=90] \arrow[r] & p_1 \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$
since $(H+kE).E=0$

Step 3. Write the image of $X \xrightarrow{|H+kE|} \mathbb{P}^*$ as X_1

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & X_1 \\ & X_0 \arrow[u, "\text{normalization}"'] \end{tikzcd}$$

$\because X$ nonsingular thus is normal

$f(E) = p$

fibres of normalization is geometrically finite

Since H is very ample

$$X\setminus E \cong \underbrace{X_1\setminus\{p_1\}}_{\text{thus is nonsingular}} \cong X_0\setminus\{p\}$$

Step 4. (Zariski's main theorem)

$f: X \to Y$ projective morphism between Noetherian integral schemes

$Y \ni y$ normal $\Longrightarrow$ $f^{-1}(y)$ connected or $f_* \mathcal{O}_X = \mathcal{O}_Y$

Thus, $f_* \mathcal{O}_X = \mathcal{O}_{X_0}$

Let X_n be subscheme of X defined by $\underline{m_p^n \mathcal{O}_X}$

pull back of m_y^n

Theorem of formal functions

$\mathcal{F}$ coherent on X

$f: X \to Y \ni y$ projective

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow & & \uparrow \\ X_n & \to & \mathrm{Spec}\, \mathcal{O}_y/m_y^n \end{array}$$

then $\left(\widehat{R^i f_* \mathcal{F}}\right)_y \cong \varprojlim H^i(X_n, \mathcal{F}_n)$

$\Longrightarrow$ $\hat{\mathcal{O}}_p \cong \varprojlim H^0(X_n, \mathcal{O}_{X_n})$

To prove X is regular, it suffices to prove $\underline{\mathcal{O}_p \text{ is regular}}$

$\Updownarrow$

$\hat{\mathcal{O}}_p$ is regular

Step 5. Prove $H^0(X_n, \mathcal{O}_{X_n}) \cong k[x,y]/(x,y)^n$

then $\hat{\mathcal{O}}_p \cong k[[x,y]]$ is regular.

$n=1$, $(X_1, \mathcal{O}_{X_1}) \cong (Y, \mathcal{O}_Y)$ & $H^0(Y, \mathcal{O}_Y) \cong k$

$n>1$ $\quad 0 \to \mathcal{J}_Y^n/\mathcal{J}_Y^{n+1} \to \mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n} \to 0$

$\mathcal{J}_Y^n/\mathcal{J}_Y^{n+1} \cong S^n(\mathcal{J}_Y/\mathcal{J}_Y^2) \cong \mathcal{O}_Y(+n)$

Conormal bundle of Y

$$\to 0 \hookrightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) \to H^0(X_{n+1}, \mathcal{O}_{X_{n+1}}) \twoheadrightarrow H^0(X_n, \mathcal{O}_{X_n}) \to \underset{\overset{\shortparallel}{0}}{H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))}$$

$n=1$. $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = kx \oplus ky$

$$H^0(X_2, \mathcal{O}_{X_2}) \underset{\text{as group}}{\cong} k \oplus kx \oplus ky \qquad x^2 = xy = y^2 = 0$$

$$\cong k[x,y]/(x,y)^2$$

$n=2$. $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$ generated by x^3, x^2y, xy^2, y^3 as group

$$\Longrightarrow \underset{\tilde{x},\tilde{y}}{H^0(X_3, \mathcal{O}_{X_3})} \cong \underset{x,y}{H^0(X_2, \mathcal{O}_{X_2})} \oplus \underset{\tilde{x}^2,\, \tilde{x}\tilde{y},\, \tilde{y}^2}{H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))} \quad \text{as vector space}$$

$\tilde{x}, \tilde{y} \longmapsto x, y$

$x^2 = xy = y^2 = 0 \Longrightarrow \tilde{x}^2, \tilde{x}\tilde{y}, \tilde{y}^2$

$$\therefore \quad H^0(X_3, \mathcal{O}_{X_3}) \cong k[x,y]/(x,y)^3$$

then rest is to prove by induction, X_0: smooth projective surface

Remark: If $Y^2 = -n$, then following the same proof X_0 would have a $\mathbb{Z}_n$-qutient singularity at p.

For instance,

$$\begin{tikzcd} K_{\mathbb{P}^1} \arrow[r, "\text{contracting } s"] \arrow[d] & \mathbb{C}^2/\mathbb{Z}_2 \\ \mathbb{P}^1 \arrow[u, bend left, "s"] & \end{tikzcd}$$

zero section $s^2 = -2$

Step 6. X is the monoidal transformation of X_0 at p.

universality of blow up $\Longrightarrow$

$$\begin{tikzcd} & X \arrow[r] \arrow[dr, bend left, dashed, "\varphi"] & X_0 \\ Y \arrow[ur, "\subseteq"] \arrow[drr, "\cong"'] & & \mathrm{Bl}_p X_0 \arrow[u] \\ & & E \arrow[u, "\subseteq"'] \end{tikzcd}$$

$\# \varphi^{-1}(x)$

lower-semi-continuous

dropped when ramified

- $\varphi|_{X \setminus Y} : X \setminus Y \xrightarrow{\cong} \mathrm{Bl}_p X_0,$
- Can choose a generic smooth curve C passing through x ($Y \ni x$) s.t. $\varphi : \underset{\substack{\cup\\ x}}{C} \xrightarrow{\cong} \underset{\substack{\cup\\ \varphi(x) \in E}}{\varphi(C)}$ smooth

then induces $\mathcal{O}_{\varphi(x)}/m_{\varphi(x)} \xrightarrow{\varphi_*} \mathcal{O}_x/m_x$

Caveat: It is important to have the map φ in advance

ex. flop

$0 \times \{t_1t_2 = t_3t_4\} \cong \mathbb{P}^1_{(u_1,v_1)} \times \mathbb{P}^1_{(u_2,v_2)} \subseteq \mathrm{Bl}_0 W \subseteq \mathbb{C}^4_{(x,y,z,w)} \times \mathbb{P}^3_{(t_1,t_2,t_3,t_4)}$

$(x,y,z,w) \sim (t_1,t_2,t_3,t_4)$

Contraction of a ruling $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$

Contraction of another ruling

$$\begin{tikzcd} & \mathrm{Bl}_0 W \arrow[dl] \arrow[d] \arrow[dr] & \\ (W_+, \mathbb{P}^1_{(u_2,v_2)}) \arrow[r, dashed, "\text{flop}"] \arrow[dr] & {} & (W_-, \mathbb{P}^1_{(u_1,v_1)}) \arrow[dl] \\ & W = \{xy - zw = 0\} \subseteq \mathbb{C}^4 & \end{tikzcd}$$

$(W_+, \mathbb{P}^1_{(u_2,v_2)}) \subseteq W_+ \times \mathbb{P}^1_{(u_2,v_2)}$ closure of the graph

$W = \{xy - zw = 0\} \subseteq \mathbb{C}^4 \xleftarrow{\text{degeneration}} \{xy - zw = \varepsilon\}$

conifold

resolved conifold $\cong T^*S^3$

$\frac{x}{z} = \frac{w}{y} = \cancel{\frac{u_2}{v_2}}$ not well-defined at 0 $\dashrightarrow \mathbb{P}^1$

$\frac{x}{w} = \frac{z}{y} = \frac{u_1}{v_1} \dashrightarrow \mathbb{P}^1$

ex. Blow up of $\mathbb{P}^2$ on generic $9-d$ points, $0 \le d \le 8$.

$X_9 \cong \mathbb{P}^2 \qquad |-K_{X_9}| \cong \mathbb{P}^{10-1} \qquad X_d = \mathrm{Bl}_p X_{d+1} \longrightarrow X_{d+1}$, $p \in X_d$

Cubics

$x^3, y^3, z^3, x^2y, xy^2$

$y^2z, yz^2, x^2z, xz^2, xyz$

very ample

$K_{\mathbb{P}^2}^2 = 9$

by induction

$K_{\tilde{X}}^2 = K_X^2 - 1$

$K_{X_d}^2 = d$ — degree

$h^0(X_d, \underline{-K_{X_d}}) = d+1 \Longrightarrow X_d \xhookrightarrow{i} \mathbb{P}^d$. $\quad d \geq 3$

generated by global sections

actually very ample

$i^*\mathcal{O}(1) = K_{X_d}$

these are called del Pezzo surfaces.

ex. When $d=9$. $\exists$ a pencil in $|-K_{\mathbb{P}^2}|$ passing through the 9 points

In other words, the 9 points are the base points of the pencil $|P|$

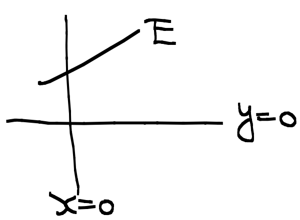
$$\begin{array}{c} Y \\ {\scriptstyle |P|}\big\downarrow \\ \mathbb{P}^1 \end{array}$$

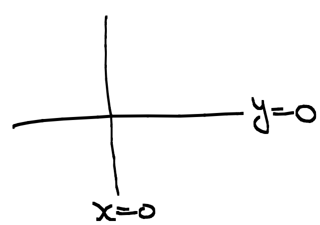
rational elliptic surface

$K_Y = -F$, F fibre

ex. blow up & cluster charts

$$(\tilde{Y} = Bl_{(0,-1)}\mathbb{C}^2,\ \tilde{D} = \tilde{\pi}D) \xrightarrow{\pi} (Y = \mathbb{C}^2, D = \{x=0\})$$





$Bl_{(0,1)}\mathbb{C}^2 \subseteq \mathbb{C}^2_{(x,y)} \times \mathbb{P}^1_{(s,t)}$ defined by $\dfrac{x}{s} = \dfrac{y+1}{t}$

$$\tilde{X} = \tilde{Y} \setminus \tilde{D} = \left\{ ((u,y),x) \in \mathbb{C}^2_{(u,y)} \times \underset{x \neq 0}{\mathbb{C}^*_x} \;\middle|\; ux = y+1 \right\} \qquad \frac{t}{s} = u$$

Claim: $\tilde{X}$ is gluing of two copies of $(\mathbb{C}^*)^2$ via cluster transformation, up to codimension two.

$$(\mathbb{C}^*)^2_1 \xrightarrow{\alpha_1} \widetilde{X} \qquad (\mathbb{C}^*)^2_2 \xrightarrow{\alpha_2} \widetilde{X}$$

$$(x_1, y_1) \longmapsto \left(\left(\frac{y_1+1}{x_1}, y_1\right), x_1\right) \qquad (x_2, y_2) \longmapsto \left((x_2^{-1}, y_2), (y_2+1)x_2\right)$$

$$\widetilde{X} = \operatorname{Im}\alpha_1 \cup \operatorname{Im}\alpha_2 \cup \underline{((\overset{u}{0}, \overset{y}{-1}), \overset{x}{0})}$$

missing a point on the exceptional divisor

$$(\mathbb{C}^*)^2_1 \overset{\alpha_2^{-1}\circ\alpha_1}{\dashrightarrow} (\mathbb{C}^*)^2_2 \qquad \text{birational}, \quad (\alpha_2^{-1}\circ\alpha_1)^*(dx_2 \wedge dy_2) = dx_1 \wedge dy_1$$

$$(x_1, y_1) \longmapsto (\underset{x_2}{\underset{\parallel}{x_1(1+y_1)}}, \underset{y_2}{\underset{\parallel}{y_1}})$$

cluster transformation

Gross-Hacking-Keel:

- Cluster varieties are blow up of toric varieties at non-toric points.
- Each blow up at a non-toric point introduce an additional cluster chart.

covers a dense subset of the exceptional divisor

Remark: The mirror of $(\widetilde{Y}, \omega)$ is given by

$$\left\{ (u, v, w) \in \mathbb{C}^2_{(u,v)} \times \mathbb{C}^*_w \;\middle|\; uv = 1 + e^{\omega(E)} w \right\},$$

also in the similar form of $\widetilde{X}$.